Coalgebraic Renormalisation as a Manifestation of Axiomatic Erosion

(Fixed-Point Flow of Differentiation–Binding Functors for QFT)

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Abstract

We recast Wilsonian renormalisation as a coalgebraic flow generated by the *differentiation–binding* endofunctor $F = \mathcal{P}_{\mathrm{f}} \circ \Delta$ on Set. A coarse–graining functor $\mathcal{C} : \mathbf{Set} \to \mathbf{Set}$ is shown to preserve finitary structure, yielding a composite $\mathcal{R} := \mathcal{P}_{\mathrm{f}} \circ \Delta \circ \mathcal{C}$ whose terminal coalgebra $(\Omega_{\mathcal{R}}, \zeta_{\mathcal{R}})$ realises *Breeze Theory*'s substrative frequency at successive length scales. Fixed points of $\mathcal{R}$ correspond to scale-invariant theories; their unavoidable undecidable observables operationalise **Axiomatic Erosion (AxE)** within quantum-field coarse-graining. The framework supplies a numerics-free, structurally complete route from category-level recursion to physical renormalisation group flows.

Contents

1 Introduction

Motivation. Renormalisation group (RG) methods form the backbone of modern quantum field theory and statistical physics, describing how physical laws evolve with scale. Traditional formulations, from Wilsonian coarse-graining to flow equations, emphasize functional recursion on coupling constants and observables. However, these approaches typically rely on analytic machinery or numerical approximations.

This paper reframes RG flow categorically: as a coalgebraic recursion generated by a structured endofunctor on **Set**. We introduce a composite functor $\mathcal{R} = \mathcal{P}_{\mathrm{f}} \circ \Delta \circ \mathcal{C}$ whose terminal coalgebra encodes the fixed-point architecture of scale-invariant theories. Each component reflects a step in the recursive structure: $\mathcal{C}$ handles spatial averaging, Δ encodes pairwise interaction structure, and $\mathcal{P}_{\mathrm{f}}$ binds these into finite relational units.

This structure matches and refines the recursive metaphysics introduced in Breeze Theory, wherein differentiation and binding underlie all emergent phenomena. By demonstrating that RG fixed points preserve undecidability across truncations, we operationalize Breeze Theory's principle of *Axiomatic Erosion*: all differentiated expressions remain logically incomplete at every expressed scale.

The result is a numerics-free, category-theoretic scaffold for renormalisation that aligns the structural logic of physics with a universal recursive ontology.

2 From Differentiation–Binding to Renormalisation

2.1 Functors Δ and $\mathcal{P}_{\mathrm{f}}$

We recall the key categorical functors previously formalized in Breeze Theory [3, 5], now adapted to serve as generative components in a renormalisation-theoretic setting.

Definition 2.1 (Unordered-pair functor Δ)**.** Let X be a set. Define

$$\Delta(X) = \{\, \{x, y\} \subseteq X : x \neq y \,\},$$

the set of all unordered distinct pairs from X. For any map $f : X \to Y$, define $\Delta(f)$ as the image of the pairing:

$$\Delta(f)(\{x, y\}) = \{f(x), f(y)\}.$$

Definition 2.2 (Finitary powerset functor $\mathcal{P}_{\mathrm{f}}$)**.** For a set X, define

$$\mathcal{P}_{\mathrm{f}}(X) = \{\, A \subseteq X : A \text{ is finite} \,\}.$$

For any function $f : X \to Y$, the action of $\mathcal{P}_{\mathrm{f}}$ is given by image:

$$\mathcal{P}_{\mathrm{f}}(f)(A) = f[A] = \{\, f(a) : a \in A \,\}.$$

Interpretation. Within the recursive notation system of Breeze Theory:

$$S(e) := \Delta \quad \text{(Excendence: Differentiation)}, \qquad S(i) := \mathcal{P}_{\mathrm{f}} \quad \text{(Incendence: Binding)},$$

and the composite substrative frequency is expressed as:

$$S(\infty) := \mathcal{P}_{\mathrm{f}} \circ \Delta.$$

2.2 Coarse–Graining as an Endofunctor

To account for scale-dependent structure in physical systems, we introduce a coarse–graining operation on configurations defined over spatial lattices. This will enable us to formulate renormalisation flows as coalgebraic constructions built atop recursive differentiation.

Definition 2.3 (Block-spin / averaging functor $\mathcal{C}$)**.** Let $\Lambda \subseteq \mathbb{R}^d$ be a regular d-dimensional lattice, and let $X \subseteq \mathbb{R}^\Lambda$ denote a configuration space over Λ. Define the *coarse–graining functor*

$$\mathcal{C}(X) := \{\, \text{block averages of elements in } X \text{ over } b^d \text{ lattice cubes} \,\}.$$

That is, for block size b, define each element of $\mathcal{C}(X)$ to be the mean value of the original configuration restricted to non-overlapping blocks of size b^d.

For a morphism $f : X \to Y$ (pointwise function between configurations), define $\mathcal{C}(f)$ as the induced map on the coarse-averaged configurations. That is, for block averages $\mathrm{avg}_b(\phi) \in \mathcal{C}(X)$, define:

$$\mathcal{C}(f)\,(\mathrm{avg}_b(\phi)) := \mathrm{avg}_b(f \circ \phi).$$

To ensure the full renormalisation functor $\mathcal{R}$ is finitary, we verify that each of its components preserves filtered colimits. The following two lemmas establish this for Δ and $\mathcal{C}$, respectively.

Lemma 2.4. *The unordered-pair functor Δ preserves filtered colimits.*

Proof. The functor Δ maps X to $\{\{x, y\} \subseteq X : x \neq y\}$, and acts on morphisms via direct image. Since filtered colimits in **Set** commute with finite limits and Δ involves finite subset formation, this functor preserves filtered colimits. □

Lemma 2.5 (Preservation of finitary structure)**.** *The coarse–graining functor $\mathcal{C}$: **Set** $\to$ **Set** preserves filtered colimits and finite limits. Hence, it respects the categorical finitary structure required for coalgebraic fixed-point constructions.*

Proof. Filtered colimits in **Set** correspond to directed unions. Coarse–graining acts on configuration spaces with finite local support (due to the bounded block size b). Hence, filtered colimits commute with block-wise averaging since each coarse output depends only on finitely many inputs.

Finite limits in **Set** include products and equalizers. Since $\mathcal{C}$ operates by local averaging (a functorial process that respects finite data flow), it commutes with these constructions as well. Thus, $\mathcal{C}$ preserves both filtered colimits and finite limits. □

Together with Δ and $\mathcal{P}_{\mathrm{f}}$, the coarse–graining functor enables the construction of a recursive RG flow over configuration spaces while retaining compatibility with the coalgebraic structure required by Breeze Theory.

2.3 The Renormalisation Endofunctor

We now define the full recursive renormalisation flow as a composition of the differentiation–binding functor with coarse–graining:

$$\mathcal{R} = \mathcal{P}_{\mathrm{f}} \circ \Delta \circ \mathcal{C} : \mathbf{Set} \longrightarrow \mathbf{Set}.$$

Proposition 2.6 (Finitarity of $\mathcal{R}$)**.** *The renormalisation endofunctor $\mathcal{R}$ is finitary.*

Proof. Recall that a functor F : **Set** $\to$ **Set** is finitary if it preserves filtered colimits. It is known that:

- $\mathcal{P}_{\mathrm{f}}$ is finitary since it preserves filtered colimits by construction,
- Δ preserves filtered colimits (see Lemma 2.4),

- $\mathcal{C}$ preserves filtered colimits (Lemma 2.5).

The Composition of a finitary functor with filtered-colimit-preserving ones is finitary. (Note: while $\mathcal{C}$ also preserves finite limits, this is not required here.)

This finitarity condition guarantees the existence of a terminal coalgebra under Barr's theorem, enabling recursive RG fixed-point construction. $\square$

3 Terminal $\mathcal{R}$–Coalgebra and Scale-Invariant Fixed Points

Given that $\mathcal{R} = \mathcal{P}_{\mathrm{f}} \circ \Delta \circ \mathcal{C}$ is finitary (Proposition 2.6), we now apply Barr's theorem to establish the existence of a unique fixed-point structure encoding recursive renormalisation flow.

Theorem 3.1 (Existence of Terminal $\mathcal{R}$-Coalgebra)**.** *Let $\mathcal{R} = \mathcal{P}_{\mathrm{f}} \circ \Delta \circ \mathcal{C}$. Then there exists a terminal coalgebra:*

$$(\Omega_{\mathcal{R}}, \zeta_{\mathcal{R}}) \quad \textit{such that} \quad \zeta_{\mathcal{R}} : \Omega_{\mathcal{R}} \xrightarrow{\sim} \mathcal{R}(\Omega_{\mathcal{R}}).$$

Proof. By Barr's fixed-point theorem [1], every finitary endofunctor $F : \mathbf{Set} \to \mathbf{Set}$ admits a terminal coalgebra. Since $\mathcal{R}$ is finitary by Proposition 2.6, the result follows. $\square$

Definition 3.2 (RG Fixed Point (Scale-Invariant Theory))**.** A coalgebra morphism

$$\phi : (\Omega_{\mathcal{R}}, \zeta_{\mathcal{R}}) \to (\Omega_{\mathcal{R}}, \zeta_{\mathcal{R}})$$

is called an *RG fixed point* if it satisfies:

$$\zeta_{\mathcal{R}} \circ \phi = \mathcal{R}(\phi) \circ \zeta_{\mathcal{R}}.$$

If $\phi = \mathrm{id}_{\Omega_{\mathcal{R}}}$, it corresponds to the canonical scale-invariant theory. More generally, any endomorphism ϕ satisfying this condition defines a self-consistent RG fixed point.

This captures the idea that recursive consistency under renormalisation is equivalent to internal invariance under scale transformation — a key insight in both physical and categorical formulations of fixed-point theory.

Theorem 3.3 (Axiomatic Erosion under $\mathcal{R}$)**.** *Let $\mathcal{Q} : \Omega_{\mathcal{R}} \to \{0, 1\}$ be any recursively enumerable predicate which is undecidable on $\Omega_{\mathcal{R}}$. Then for some $n \in \mathbb{N}$, its restriction $\mathcal{Q} \circ \iota_n$ remains undecidable on the finite-scale truncation:*

$$\iota_n : \mathcal{R}^n(1) \hookrightarrow \Omega_{\mathcal{R}}.$$

Sketch. As in Theorem 5.1 of [5], we can express $\Omega_{\mathcal{R}}$ as a colimit of finite stages:

$$\Omega_{\mathcal{R}} = \varinjlim_{n} \mathcal{R}^n(1),$$

where $\mathcal{R}^n(1)$ is the n-fold application of $\mathcal{R}$ to the terminal object $1 \in \mathbf{Set}$.

If $\mathcal{Q}$ is undecidable on $\Omega_{\mathcal{R}}$, then there must exist some finite stage $\mathcal{R}^n(1)$ such that $\mathcal{Q} \circ \iota_n$ remains undecidable. Thus, recursive undecidability persists under finite-scale approximations.

This operationalises **Axiomatic Erosion**: no finite-scale instantiation of $\mathcal{R}$ can fully resolve all substrative observables. $\square$

4 Connection to Physical Renormalisation Group

The coalgebraic formulation of $\mathcal{R} = \mathcal{P}_{\mathrm{f}} \circ \Delta \circ \mathcal{C}$ gives a categorical backbone to the standard physical intuition behind the renormalisation group (RG). In physics, RG describes how a physical system appears when probed at different length scales, especially how coupling constants and effective interactions change under coarse-graining. Here, we reinterpret such flows as coalgebraic recursion, structured by Breeze Theory's differentiation–binding loop.

Interpretation

- Δ: Encodes pairwise interactions (e.g., nearest-neighbor coupling).
- $\mathcal{P}_{\mathrm{f}}$: Captures bounded coherence — finite, stable groupings of interaction terms.
- $\mathcal{C}$: Implements spatial averaging or block-spin transformation, grouping microscopic degrees of freedom.

Composed as $\mathcal{R}$, these functors describe how recursive relational structure evolves under spatial scale transformations. The fixed points of $\mathcal{R}$ correspond to scale-invariant structures such as those found at critical points in phase transitions.

Example 4.1 (Scalar ϕ^4 Lattice Field Theory)**.** Identify the coupling constant u_n with the image of the configuration class $[\phi_n] \in \Omega_{\mathcal{R}}$ under an evaluation functional:

$$\ell : \Omega_{\mathcal{R}} \to \mathbb{R}, \quad \ell([\phi_n]) = u_n.$$

Consider the RG recursion relation:

$$u_{n+1} = b^{d-4} u_n - c\, u_n^2 + \dots$$

where u_n is the coupling constant at the n-th scale, b is the rescaling factor, and d is the spacetime dimension.

In our framework, $\mathcal{R}$ captures this recursion abstractly. The coarse-graining functor $\mathcal{C}$ collapses local field configurations into averaged blocks, Δ maps pairwise interactions between blocks, and $\mathcal{P}_{\mathrm{f}}$ binds them into finite interaction sets. The limit object $\Omega_{\mathcal{R}}$ encodes the structure-preserving pattern across all scales.

Thus, fixed points of $\mathcal{R}$ correspond to $u_n = u^*$ such that the structure of $\Omega_{\mathcal{R}}$ remains invariant, matching the Wilson-Fisher interpretation of scale-invariant field theories.

5 Alignment with Breeze Theory

(a) The object $\Omega_{\mathcal{R}}$ generalizes the substrative fixed point Ω (defined in [3]) by introducing scale-dependent recursion via coarse-graining. This captures how recursive structure unfolds through physically meaningful scale flows.

(b) RG fixed points operationalize Breeze Theory's principle that recursion governs structure at all levels. The persistence of undecidable observables across truncations (Theorem 3.3) manifests Axiomatic Erosion in physical systems — no level of analysis escapes incompleteness.

(c) The composite functor structure $\mathcal{R} = \mathcal{P}_{\mathrm{f}} \circ \Delta \circ \mathcal{C}$ aligns directly with Breeze Theory's recursive notation: $S(i) = \mathcal{P}_{\mathrm{f}}$, $S(e) = \Delta$, with $\mathcal{C}$ acting as a context-sensitive adjustment layer or local renexial gradient (as defined in Breeze Notation)[4] across scales.. The substrative frequency $S(\infty)$ emerges as a scale-invariant recursive binding function.

6 Outlook

This formalization opens several directions for expansion:

- **Constructive QFT:** Use the coalgebraic RG flow to reinterpret quantum field theory models as structured recursive systems, especially ϕ^4 theory and Ising models.
- **Monte Carlo Simulation Interfaces:** Translate $\mathcal{R}$ into a categorical wrapper for probabilistic lattice simulation methods, allowing theoretical predictions about observable erosion under truncation.
- **Categorical RG in Prob or Meas:** Extend $\mathcal{R}$ to stochastic settings, enabling probabilistic models of axiomatic erosion under uncertainty.
- **Substraeternum Refinement:** Study the relationship between $\Omega_{\mathcal{R}}$ and the Substraeternum equation $\aleph_\delta = f_\infty(\delta) = \infty(\delta(\infty))$ to determine whether the recursion fixed by RG flow is a proper subset of the global recursive substrate.
- **Recursive Universality Classifications:** Explore whether different $\mathcal{R}$ functor compositions yield structurally distinct recursion classes — a new lens on universality classes.

References

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Proof constructed in eternal yet fluid appendix to **Breeze Theory: A Foundational Framework for Recursive Reality**.

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